A Counterexample to Kippenhahn's Conjecture on Hermitian Pencils

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ABSTRACT

A counterexample is constructed to a conjecture of Kippenhahn (*Math. Nachr.* 6:193–228 (1951–52)). A pair of Hermitian 8×8 matrices H, K is found such that (1) H, K generate $M_8(\mathbf{C})$ and (2) the minimal polynomial of the pencil $xH+yK$ has degree 4. Recent work of H. Shapiro [e.g. *Linear Algebra Appl.* 43:201–221 (1982)] has established the conjecture for $n\times n$ matrices $n\leqslant 5$.

Let H, K be $n\times n$ (complex) Hermitian matrices, and let $f(x,y,z)=\det(xH+yK-zI)\in\mathbf{C}[x,y,z]$ be the characteristic polynomial of the pencil $xH+yK$. Kippenhahn [2] has conjectured that if $f(x,y,z)$ has a repeated factor in the polynomial ring $\mathbf{C}[x,y,z]$, then H, K are (simultaneously) unitarily similar to direct sums $H_1\oplus H_2$, $K_1\oplus K_2$ with $H_i, K_i\in M_{n_i}(\mathbf{C})$ for some n_i with $1\leqslant n_i<n$, $i=1,2$ [or, equivalently, using Burnside's theorem, H, K do not generate $M_n(\mathbf{C})$]. Kippenhahn verified the conjecture whenever the degree of the minimal polynomial of $xH+yK$ has degree 1 or 2. In recent work [4, 5], Shapiro has obtained a number of results which support the conjecture. In particular, she shows that it holds if $n\leqslant 5$. In this note, however, we show that the conjecture fails in general, by constructing a counterexample with $n=8$. Using different methods, Waterhouse [6] has independently found counterexamples to the conjecture.

Our construction is based on the following observation: A matrix A is nonderogatory if and only if the only matrices which commute with A are the polynomials in A. If H, K are real symmetric matrices such that $xH+yK$ commutes with a nonsymmetric matrix in the matrix ring $M_n(\mathbf{C}[x,y])$, then $xH+yK$ is derogatory as a matrix in $M_n(F)$, where F is the field $\mathbf{C}(x,y)$. Hence (using Gauss's lemma), $f(x,y,z)$ has a repeated factor in $\mathbf{C}[x,y,z]$, so the pair (H,K) satisfies the hypotheses of Kippenhahn's conjecture.

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EXAMPLE. Let

$$A=\begin{bmatrix} P & X \\ -X' & Q \end{bmatrix}, \qquad B=\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix},$$

where

$$P=\begin{bmatrix} 0 & -1 & 3 & -6 \\ 1 & 0 & -6 & -3 \\ -3 & 6 & 0 & 1 \\ 6 & 3 & -1 & 0 \end{bmatrix}, \qquad Q=\begin{bmatrix} 0 & -2 & -2 & 6 \\ 2 & 0 & 6 & 2 \\ 2 & -6 & 0 & 2 \\ -6 & -2 & -2 & 0 \end{bmatrix},$$

$$X=\begin{bmatrix} -1 & -1 & 5 & -7 \\ -1 & 1 & -7 & -5 \\ -1 & 13 & 1 & -1 \\ 13 & 1 & -1 & -1 \end{bmatrix}, \qquad U=\begin{bmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

(where $'$ denotes transpose).

Let $H = A^2$ and $K = AB + BA$.

Note first that $U^2 = -4I$, so $B^2 = -4I$. Hence $xA + yB$ commutes with $xH + yK$. Note that A, B are (real) skew-symmetric matrices and that H, K are symmetric. Hence as a matrix in $M_8(\mathbf{C}(x, y))$, $xH + yK$ is derogatory, so the pair H, K satisfies the hypotheses of Kippenhahn's conjecture. Thus in order to invalidate the conjecture, it is sufficient to show that the algebra $\mathcal{A}$ generated (over $\mathbf{C}$) by H, K is $M_8(\mathbf{C})$.

To carry this out we first perform the calculations

$$K=\begin{bmatrix} -4I_2 & 0 & 0 & 0 \\ 0 & 4I_2 & 0 & 0 \\ 0 & 0 & -8I_2 & 0 \\ 0 & 0 & 0 & 8I_2 \end{bmatrix},$$

$$H=\begin{bmatrix} H_1 & H_2 \\ H_2' & H_3 \end{bmatrix},$$

where

$$H_1=\begin{bmatrix} -122 & 0 & 12 & 18 \\ 0 & -122 & -6 & -12 \\ 12 & -6 & -218 & 0 \\ 18 & -12 & 0 & -218 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} -30 & 18 & 26 & 10 \\ -16 & -28 & 20 & -16 \\ 44 & 8 & 24 & 12 \\ -2 & -34 & -10 & 22 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} -216 & 0 & -12 & -8 \\ 0 & -216 & -8 & 36 \\ -12 & -8 & -120 & 0 \\ -8 & 36 & 0 & -120 \end{bmatrix}.$$

For $Y \in M_8(\mathbf{C})$, write $Y = (Y_{ij})$ as a block matrix where $Y_{ij} \in M_2(\mathbf{C})$. We write $\hat{Y}_{ij}$ for the matrix whose (i, j) block is Y_{ij} and all whose other blocks are zero. Since -4, 4, -8, and 8 are distinct, we can find polynomials f_i $(i = 1,2,3,4)$ such that $f_i(K)$ has an I_2 in the ith diagonal block and zeros everywhere else. Since $K \in \mathcal{C}$, we thus see that if $Y \in \mathcal{C}$, so does $\hat{Y}_{ij} = f_i(K)Yf_j(K)$ $(i, j = 1,2,3,4)$. Note that

$$(\hat{H}_{12}\hat{H}_{21})_{11} = \begin{pmatrix} 468 & -288 \\ -288 & 180 \end{pmatrix} = M, \quad \text{say,}$$

and that

$$(H_2 H_2')_{11} = \begin{pmatrix} 2000 & 336 \\ 336 & 1696 \end{pmatrix} = N, \quad \text{say.}$$

Since M, N are real symmetric and do not commute, they generate $M_2(\mathbf{C})$. So $\mathcal{C}$ contains $\hat{X}_{11}$ (X_{11} arbitrary). Next,

$$(\hat{H}_{21}\hat{H}_{12})_{22} = \begin{pmatrix} 180 & 288 \\ 288 & 468 \end{pmatrix} \quad \text{and} \quad (H_2 H_2')_{22} = \begin{pmatrix} 2720 & -336 \\ -336 & 1744 \end{pmatrix},$$

so, repeating the argument, we find that $\mathcal{C}$ contains $\hat{X}_{22}$ (X_{22} arbitrary). Noting now that H_{12}, H_{21} have none of their entries zero, we conclude that $\mathcal{C}$ contains

$$\left\{ \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \middle| Y \in M_4(\mathbf{C}) \right\}.$$

On considering

$$\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} H \quad \text{and} \quad H \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}$$

we see that $\mathcal{C}$ contains

$$\begin{bmatrix} 0 & YH_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ H_2'Y & 0 \end{bmatrix}$$

respectively, for all $Y \in M_4(\mathbf{C})$. On multiplying these, we see that $\mathcal{Q}$ contains

$$\begin{bmatrix} 0 & 0 \\ 0 & H_2'YH_2 \end{bmatrix}$$

for all $Y \in M_4(\mathbf{C})$. Hence $\mathcal{Q}$ contains

$$\begin{bmatrix} M_4(\mathbf{C}) & M_4(\mathbf{C})H_2 \\ H_2'M_4(\mathbf{C}) & H_2'M_4(\mathbf{C})H_2 \end{bmatrix}.$$

But H_2 is nonsingular, so $\mathcal{Q} = M_8(\mathbf{C})$, as claimed.

Since for x, y real, $xA + yB$ is real skew-symmetric, its eigenvalues occur in complex conjugate pairs $\pm ia$ (a real), so the eigenvalues of $(xA + yB)^2 = x^2H + xyK - 4y^2I$ occur with even multiplicity. This implies that the minimal polynomial of $xH + yK$ has degree at most 4. By [5], it thus follows that it has degree exactly 4.

The work of Friedland [1] (in which Kippenhahn's conjecture is recorded as Conjecture 11.17) contains many results on the simultaneous reducibility of a pair of matrices, and the work of Raykhsteyn [3] contains results on the simultaneous reducibility of Hermitian pencils.

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